# National Institute of Technology, Hamirpur(H.P.)

**End Term Examination (December 2020)**
[Class: M.Sc. (Mathematics & Computing) (3rd Semester)]
Title of course: Functional Analysis
Course Code: MA-631

**Time: Two Hours** **Maximum Marks: 50**

**Note:** All questions are compulsory.

Q1 Do the followings: (Each has 2 marks)

(i) Give an example of a metric space which is not a normed space.

(ii) Let $X$ be an inner product space. If $\langle x, u\rangle = \langle x, v\rangle$ for all $x \in X$, show that $u = v$.

(iii) Define a Hilbert adjoint operator, self adjoint operator and normal operator.

(iv) Does the $\|x\| = x^2,\ x \in \mathbb{R}$, define a norm on $\mathbb{R}$? Justify your answer.

(v) Let $T^n$ be a contraction on a Banach space $X$ for some $n > 1$. Show that $T$ has a unique fixed point in $X$.

Q2 State and prove Closed graph theorem. (10 marks)

Q3 Let $H$ be a complex Hilbert space and $T$ be a positive operator on $H$. Show that $I + T$ is invertible on $H$. (8 marks)

Q4 Let $X$ and $Y$ be normed space defined on same scalar field. If a linear operator $T\colon X \to Y$ is continuous on $X$, then show that it is bounded. (6 marks)

Q5 State and prove Schwartz inequality. (5 marks)

Q6 Let $N$ be a normed space and $x_0$ be a non-zero vector in $N$. Using Hahn-Banach theorem, show that there is a bounded linear functional $f_0$ on $N$ such that

$$f_0(x_0) = \|x_0\| \quad \text{and} \quad \|f_0\| = 1.$$

(4 marks)

Q7 Let $T\colon C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ be defined by

$$(Tx)(t) = \int_0^t x(\tau)\, d\tau.$$

Find $R(T)$ and $T^{-1}\colon R(T) \to C([0,1],\mathbb{R})$. Is $T^{-1}$ linear and bonded? (4 marks)

Q8 Let $X = l_2(\mathbb{R})$. Choose a fixed $\alpha = (\alpha_i)_{i=1}^{\infty} \in X$ and define a map $f\colon X \to \mathbb{R}$ as $f(x) = \sum_{i=1}^{\infty} \alpha_i x_i$, $x = (x_i)_{i=1}^{\infty} \in X$. Find the value of $\|f\|$. (3 marks)

****End of the paper****